Partial Derivatives: - Euler's Theorem

Let $z = f(x, y)$, Take $x = a$, $y = b$, so that the value of $z = f(a,b)$, we keep the value of y fixed at b; we only change the value of ~~x~~ x from $x = a$ to $x = a+h$. The new value of $z = f(a+h, b)$.

If now

$$\lim_{h \to 0} \frac{f(a+h, b) - f(a,b)}{h} \text{ exists}$$

then we call it the partial derivative of $f(x,y)$ w.r.to x at the point (a,b) and denoted by

$$\left(\frac{\partial f}{\partial x}\right)_{x=a,\, y=b} \quad \text{or} \quad f_x(a,b)$$

Similarly, keeping x fixed at a, we give an increment k to y so that y changes from $y = b$ to $y = b + k$, then

$$\lim_{h \to 0} \frac{f(a, b+k) - f(a,b)}{h}, \text{ if it exists.}$$

is called the partial derivative of $f(x,y)$ w.r. to y at the point (a,b) and denoted $\left(\frac{\partial f}{\partial y}\right)_{x=a,\, y=b}$ or $f_y(a,b)$

* ~~∂ del~~ ∂ means del

∂f read as ~~∂~~ delf

∂x " " del x

∂y " " del y

Rule 1 :- To find $\frac{\partial f}{\partial x}$, take the ordinary derivative of $f(x,y)$ w.r.t. x as if y treated as Constant

i,e if $f(x,y) = x^2 + xy + y^2$, then $\frac{\partial f}{\partial x} = 2x + y$

Rule 2 :- To find $\frac{\partial f}{\partial y}$, take the ordinary derivative of $f(x,y)$ w.r. to y if x treated as Constant

i,e if $f(x,y) = x^2 + xy + y^2$, then $\frac{\partial f}{\partial y} = x + 2y$

Partial Derivatives of Higher order

i) $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}(x,y) = f_{xx}$

i.e if $f(x,y) = x^2 + xy + y^2$

$$\frac{\partial f}{\partial x} = f_x = 2x + y \quad [y \text{ treated as constant})$$

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} = \frac{\partial}{\partial x}(2x+y) = 2 \quad [y \text{ treated as constant})$$

ii) $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{yx}$

if $f(x,y) = x^2 + xy + y^2$

$$\frac{\partial f}{\partial x} = f_x = 2x + y \quad (y \text{ treated as constant})$$

$$\frac{\partial^2 f}{\partial y \partial x} = f_{yx} = \frac{\partial}{\partial y}(2x+y) = 1 \quad (x \text{ treated as constant})$$

iii) $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{xy}$

if $f(x,y) = x^2 + xy + y^2$

$$\frac{\partial f}{\partial y} = f_y = x + 2y \quad (x \text{ treated as constant})$$

$$\frac{\partial^2 f}{\partial x \partial y} = f_{xy} = \frac{\partial}{\partial x}(x + 2y) = 1 \quad (y \text{ treated as constant})$$

iv) $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial^2 y} = f_{yy}$

if $f(x,y) = x^2 + xy + y^2$

$$\frac{\partial f}{\partial y} = x + 2y \quad (x \text{ treated as constant})$$

$$\frac{\partial^2 f}{\partial y^2} = f_{yy} = \frac{\partial}{\partial y}(x + 2y) = 2 \quad (x \text{ treated as constant})$$

Here $\frac{\partial^2 f}{\partial x \partial y}$ or $\frac{\partial^2 f}{\partial y \partial x}$ are called Mixed Partial derivatives

Example :-

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $x=3$, $y=2$, $f(x,y) = x^2+y^2+3xy$

Ans:- $f(x,y) = x^2+y^2+3xy$

At $x=3$, $y=2$

$\frac{\partial f}{\partial x} = 2x+3y \qquad \frac{\partial f}{\partial x} = 2\cdot 3 + 3\cdot 2 = 12$

$\frac{\partial f}{\partial y} = 2y+3x \quad " \quad \frac{\partial f}{\partial x} = 2\cdot 2 + 3\cdot 3 = 13$

2. If $f(x,y) = e^{x+2y}$, Then show that $\frac{\partial f}{\partial x} = f(x,y)$, $\frac{\partial f}{\partial y} = 2f(x,y)$

Solⁿ :- $f(x,y) = e^{x+2y}$

$$\frac{\partial f}{\partial x} = e^{x+2y}.1 = e^{x+2y} = f(x,y)$$

$$\frac{\partial f}{\partial y} = e^{x+2y}\cdot\frac{\partial}{\partial y}(x+2y) = e^{x+2y}.2 = 2e^{x+2y} = 2f(x,y)$$

3. If $f(x,y) = 2x^3 - 11x^2y + 3y^3$; then show that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3f(x,y)$

Solⁿ.

$f(x,y) = 2x^3 - 11x^2y + 3y^3$

$\frac{\partial f}{\partial x} = 6x^2 - 22xy$

$\frac{\partial f}{\partial y} = -11x^2 + \cancel{6y}\, 9y^2$

So L.H.S $= x\frac{\partial f}{\partial x} + y\cdot\frac{\partial f}{\partial y}$

$= x(6x^2-22xy) + y(-11x^2+9y^2)$

$= 6x^3 - 22x^2y + -11x^2y + 9y^3$

$= 6x^3 - 33x^2y + 9y^3$

$= 3(2x^3 - 11x^2y + 3y^3)$

$= 3f(x,y) =$ R.H.S.

4. Find f_{yx} for the function $f(x,y) = x^3+y^3+2xy^2$

Soln.

$$f(x,y) = x^3+y^3+2xy^2$$

$$f_x = \frac{\partial f}{\partial x} = 3x^2+2y^2$$

$$f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(3x^2+2y^2)$$

$$= 4y$$

5. $z = \frac{x}{x^2+y^2}$, then proved that $\frac{\partial z}{\partial x} = \frac{y^2-x^2}{(y^2+x^2)^2}$, $\frac{\partial z}{\partial y} = -\frac{2xy}{(x^2+y^2)^2}$

Soln.

$$z = \frac{x}{x^2+y^2}$$

$$\frac{\partial z}{\partial x} = \frac{(x^2+y^2)\frac{\partial}{\partial x}\cdot x - x\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\cancel{\frac{\partial z}{\partial y} = \frac{x}{\quad}}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = x\cdot\frac{\partial}{\partial y}(x^2+y^2)^{-1}$$

$$= x\cdot(-1)\cdot(x^2+y^2)^{-2}\cdot 2y$$

$$= -\frac{2xy}{(x^2+y^2)^2}$$

Homogeneous Functions:- 6

Defination:- If a function $f(x,y)$ of two independent variables x & y can be written in the form $x^n\phi(y/x)$ or $y^n\phi(\frac{x}{y})$, we say that $f(x,y)$ is a homogeneous function in x and y of degree n

i.e $z = \frac{x^2y^2}{x+y} = \frac{x^4(\frac{y^2}{x^2})}{x(1+y/x)} = x^3 . \frac{(y/x)^2}{1+y/x} = x^3 \times$ a function of y/x

So z is a homogeneous function in x and y of degree ~~2~~ 3

Euler's Theorem on Homogeneous function of Two variables

If $z = f(x,y)$ be a homogeneous function of x and y of degree n having partial derivatives f_x and f_y then

$$x.f_x + y f_y = nf$$

or $$x.\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n f(x,y)$$

Example:-

1. verify Eulers Theorem if ~~$z = x^4 - x^3y + 2x^2y^2 - xy^3$~~
$z = x^3y^2 + x^2y^3$

Soln.

$$z = x^3y^2 + x^2y^3$$
$$= x^5\left(\frac{y^2}{x^2} + \frac{y^3}{x^3}\right)$$
$$= x^5\phi(y/x)$$

Thus ~~$z(x,y)$~~ is a homogeneous function in x and y of degree 5.

To verify Euler's theorem we have to prove that—

$$x.\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 5z$$

$$z = x^3y^2 + x^2y^3$$

$$\frac{\partial z}{\partial x} = 3x^2y^2 + 2xy^3$$

$$\frac{\partial z}{\partial y} = 2x^3y + 3x^2y^2$$

$$L.H.S = x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y}$$

$$= x(3x^2y^2 + 2xy^3) + y(2x^3y + 3x^2y^2)$$

$$= 3x^3y^2 + 2x^2y^3 + 2x^3y^2 + 3x^2y^3$$

$$= 5x^3y^2 + 5x^2y^3$$

$$= 5(x^3y^2 + x^2y^3)$$

$$= 5z = R.H.S$$

2. Verify Euler's Theorem if $f(x,y) = -\frac{x+y}{\sqrt{x}+\sqrt{y}}$

<u>Soln.</u> $$f(x,y) = -\frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$= -\frac{x(1+y/x)}{\sqrt{x}(1+\sqrt{y}/\sqrt{x})}$$

$$= -\sqrt{x}\,\phi(y/x)$$

Thus $f(x,y)$ is a homogeneous function in x and y of degree $\frac{1}{2}$

To verify Euler's Theorem we have to prove that—

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = \frac{1}{2}f(x,y)$$

$$f(x,y) = -\frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$\frac{\partial f}{\partial x} = -\frac{(\sqrt{x}+\sqrt{y})\cdot 1 - (x+y)\cdot\frac{1}{2}\cdot x^{-\frac{1}{2}}}{(\sqrt{x}+\sqrt{y})^2}$$

$$= -\frac{2\sqrt{x}(\sqrt{x}+\sqrt{y}) - (x+y)}{2\sqrt{x}(\sqrt{x}+\sqrt{y})^2}$$

$$\frac{\partial f}{\partial y} = -\frac{(\sqrt{x}+\sqrt{y})\cdot 1 - (x+y)\cdot\frac{1}{2}\cdot y^{-\frac{1}{2}}}{(\sqrt{x}+\sqrt{y})^2}$$

$$= -\frac{2\sqrt{y}(\sqrt{x}+\sqrt{y}) - (x+y)}{2\sqrt{y}(\sqrt{x}+\sqrt{y})^2}$$

$$L.H.S = x\cdot\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$$

$$= x\cdot -\frac{2\sqrt{x}(\sqrt{x}+\sqrt{y}) - (x+y)}{2\sqrt{x}\,(\sqrt{x}+\sqrt{y})^2} + y\cdot -\frac{2\sqrt{y}(\sqrt{x}+\sqrt{y}) - (x+y)}{2\sqrt{y}(\sqrt{x}+\sqrt{y})^2}$$

$$= -\frac{2x(\sqrt{x}+\sqrt{y}) - \sqrt{x}(x+y)}{2(\sqrt{x}+\sqrt{y})^2} - \frac{2y(\sqrt{x}+\sqrt{y}) - \sqrt{y}(x+y)}{2(\sqrt{x}+\sqrt{y})^2}$$

$$= \frac{-2x(\sqrt{x}+\sqrt{y}) + \sqrt{x}(x+y) - 2y(\sqrt{x}+\sqrt{y}) + \sqrt{y}(x+y)}{2(\sqrt{x}+\sqrt{y})^2}$$

$$= \frac{-2(\sqrt{x}+\sqrt{y})(x+y) + (x+y)(\sqrt{x}+\sqrt{y})}{2(\sqrt{x}+\sqrt{y})^2}$$

$$= -\frac{(\sqrt{x}+\sqrt{y})(x+y)}{2(\sqrt{x}+\sqrt{y})^2}$$

$$= -\frac{1}{2}\cdot\frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$= -\frac{1}{2} f(x,y) = R.H.S$$

3. Verify Euler's Theorem if $f = \frac{x+y}{x-y}$

Soln.

$$f = \frac{x+y}{x-y}$$

$$= \frac{x(1+y/x)}{x(1-y/x)}$$

$$= x^0\cdot\frac{(1+y/x)}{(1-y/x)} = x^0\phi\left(\frac{y}{x}\right)$$

Thus f is a homogeneous function in x and y of degree 0

To verify Euler's theorem, we have to prove that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial x} = 0$$

$$f = \frac{x+y}{x-y}$$

$$\frac{\partial f}{\partial x} = \frac{(x-y)\cdot 1 - (x+y)\cdot 1}{(x-y)^2}$$

$$= \frac{\cancel{(x-y)}\, x-y-x-y}{(x-y)^2}$$

$$= \frac{-2y}{(x-y)^2}$$

$$\frac{\partial f}{\partial y} = \frac{(x-y)\cdot 1 - (x+y)\cdot -1}{(x-y)^2}$$

$$= \frac{x-y+x+y}{(x-y)^2}$$

$$= \frac{2x}{(x-y)^2}$$

$$L.H.S = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$$

$$= x\cdot\frac{-2y}{(x-y)^2} + y\cdot\frac{2x}{(x-y)^2}$$

$$= -\frac{2xy}{(x-y)^2} + \frac{2xy}{(x-y)^2}$$

$$= 0 = R.H.S$$

4. Find $\frac{dy}{dx}$ from the following relation by using the implicit function. $x^4 - x^2y^2 + y^4 = 80$

<u>Soln.</u> By defination we know that $\frac{dy}{dx} = -\left[\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y}\right]$

Let $f(x,y) = x^4 - x^2y^2 + y^4$

$$\frac{\partial f}{\partial x} = 4x^3 - 2xy^2$$

$$\frac{\partial f}{\partial y} = -2x^2y + 4y^3$$

$$\frac{dy}{dx} = -\left[\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y}\right]$$

$$= -\frac{4x^3 - 2xy^2}{-2x^2y + 4y^3}$$

$$= \frac{2x(2x^2 - y^2)}{2y(x^2 - 2y^2)}$$

$$= \frac{x(2x^2 - y^2)}{y(x^2 - 2y^2)}$$